An Intelligent Method for the Control of Magnitude of Parabolic-like Transmission Error of a Pair of Gears

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Abstract—Traditionally, the control of magnitude of parabolic-like transmission error of a pair of gears requires a lot of time-consuming trial-and-error manual procedures. To advance design efficiency, this paper has proposed an intelligent method to control efficiently and accurately the magnitude of parabolic-like transmission error. The intelligent method is devised based on the development of a system of governing equations under the conditions of tooth contact and constraints for magnitude of parabolic-like transmission error. Design parameters required to be determined are transformed into roots of the system of equations. Just applying Newton’s root finding method, parameters needed to be designed are obtained automatically and efficiently. To show how to apply the proposed method, a pair of external gears composed of an involute gear and a circular-arc gear is adopted to be an example. The gear pair is verified numerically the magnitude of parabolic-like transmission error is really controlled.

Keywords—parabolic-like transmission error; tooth contact analysis; theory of gearing; circular-arc gear

I. INTRODUCTION

Ideally, a circular gear pair should have a constant ratio of velocity. However, in reality, due to many uncontrollable factors such as manufacturing errors, assembly errors, and elastic deformation of materials, the ratio of velocity cannot keep constant. The real output angle minus the theoretical output angle is defined as transmission error. For gears without any modification, the transmission error is a linear function. Thus the motion function is not a continuous function. Linear type of transmission error has many drawbacks such as serious impact of tooth faces, high level of vibration and noise, and significant reduction of lifetime. To mend the shortcomings of linear transmission error, Litvin [1] proposed to give the gear pair a pre-designed parabolic transmission error to absorb the linear one. Parabolic transmission error causes a continuous motion function and therefore reduces a large amount of gear noise. Based on Litvin’s methodology, many researchers have proposed many new types of gear pairs. For example, Litvin and Lu [2] proposed a reshaping method for a double circular-arc helical gear pair so that the gear pair has a pre-designed parabolic transmission error. Litvin and Kim [3] proposed a reshaping method for a spur gear pair to obtain a pre-designed parabolic transmission error. Seol and Litvin [4] and De Donno and Litvin [5] proposed reshaping methods for single enveloping cylindrical worm gears to obtain pre-designed parabolic functions of transmission error. Litvin et al. [6] proposed modified geometry for face-gear drives based on application of a shaper that is conjugated to a parabolic rack-cutter. Litvin et al. [7] proposed new geometry of face worm gear drives with conical and cylindrical worms based on application of head-cutters or head grinding tools. Lee and Chen [8] proposed using a pair of blade-disks to generate a cylindrical gear pair so that the function of transmission error is a parabolic one. Litvin et al. [9] reported a new geometry of face-gear drive with helical pinion. Lee et al. [10] proposed a face-gear drive with parabolic amplitude control of kinematic errors based on application of a third order polynomial profile of rack of pinion. Lee [11] proposed a cylindrical crown gear drive with a controllable fourth order polynomial function of transmission error.

The main purpose of this paper is to introduce an intelligent method to control the magnitude of parabolic-like transmission error efficiently and accurately. Traditionally, gear design engineers need to spend a lot of trial-and-error time to adjust design parameters and to check whether the magnitude of transmission error reaches the anticipated value or not. Now, the iterative tasks can be avoided with the intelligent method proposed in this paper. This method transforms the design parameters into roots of a system of governing equations. The system of equations is created based on the conditions of tooth contact and the constraint of magnitude of parabolic-like transmission error. By using the Newton’s root finding method to solve the roots of the system of equations, the values of design parameters can be obtained without any manual trial-and-error procedure. The intelligent method can be applied to two or three dimensional gear pairs. To explain the flow of creating the system of governing equations, a pair of external gears composed of an involute gear and a circular-arc gear is introduced as an example. The function between the magnitude of parabolic-like transmission error and the radius of the circular-arc gear is analyzed and found to be a nonlinear curve.
II. DEFINITION OF MAGNITUDE OF PARABOLIC-LIKE TRANSMISSION ERROR

First, the meaning of magnitude of parabolic-like transmission error is clearly defined. Fig. 1 shows a pair of meshing surfaces $\Sigma_1$ and $\Sigma_2$, where $\Sigma_1$ represents the tooth surface of the input gear while $\Sigma_2$ represents the tooth surface of the output gear. Orientation of gear axes is invariant. Angular displacements of the input and the output gears are denoted by $\phi_1$ and $\phi_2$, respectively. Ideally, parameters $\phi_1$ and $\phi_2$ should obey the following conditions:

$$\phi_1 = \left( \frac{N_1}{N_2} \right) \phi_2$$

(1)

where $N_1$ and $N_2$ denote the numbers of teeth of the input and the output gears, respectively.

Affected by unavoidable interference factors such as elastic deformation of materials, assembly errors, and manufacturing errors, the angular displacement of the output gear cannot obey Eq. (1). The real angular displacement minus the ideal angular displacement of the output gear is defined as the transmission error $\phi_2^{\Delta}$ as follows:

$$\phi_2^{\Delta} = \phi_2 - \left( \frac{N_1}{N_2} \right) \phi_1$$

(2)

Without any reshaping modification, the transmission error of a traditional pair of gears, $\phi_2^{\Delta}$ is a linear function, causing a discontinuous motion function. Thus serious impact of tooth faces and high level of noise and vibration happen. For solving the drawbacks of linear transmission error, Livtin [1] proposed to apply purposely a pre-designed parabolic transmission error to the gear pair to absorb the linear error.

Fig. 2 shows the motion function composed of three parabolic-like transmission errors. The middle parabolic-like curve represents the transmission error of the pair of teeth that are observed now. The left-side and right-side parabolic-like curves denote the transmission errors of the former and the latter pairs of teeth. The middle parabolic-like curve intersects with the left one and the right one at $\phi_1 = \phi_1^{(a)}$ and $\phi_1 = \phi_1^{(b)}$, respectively. Referring to Fig. 1 again, tooth surface $\Sigma_1$ contacts tooth surface $\Sigma_2$ at point $A$ when $\phi_1 = \phi_1^{(a)}$, at point $m$ when $\phi_1 = \phi_1^{(m)}$, and at point $B$ when $\phi_1 = \phi_1^{(b)}$. At point $m$, $\phi_1^{(m)} = 0$ and $\Delta \phi_2 = 0$. At points $A$ and $B$, transmission errors are

$$\Delta \phi_2 = -\xi$$

(3)

Here, we defined $\xi$ to be the magnitude of parabolic-like transmission error. The value of $\xi$ varies according to the level of tooth modification. In the next section, we will discuss how to create a system of governing equations to control the value of $\xi$.

III. GOVERNING EQUATIONS FOR THE CONTROL OF MAGNITUDE OF PARABOLIC-LIKE TRANSMISSION ERROR

Curvilinear coordinates of input gear’s tooth surface $\Sigma_1$ are assumed to be $u_1$ and $v_1$. Curvilinear coordinates of output gear’s tooth surface $\Sigma_2$ are assumed to be $u_2$ and $v_2$. Coordinate systems $S_1$ and $S_2$ are applied to connect rigidly to $\Sigma_1$ and $\Sigma_2$, respectively. Represented in $S_1$, position vector and unit normal vector of $\Sigma_1$ are $r_1(u_1,v_1)$ and $n_1(u_1,v_1)$, respectively. Represented in $S_2$, position vector and unit normal vector of $\Sigma_2$ are $r_2(u_2,v_2)$ and $n_2(u_2,v_2)$, respectively. A stationary coordinate system $S_f$ is applied to the frame. As tooth surface $\Sigma_1$ drives tooth surface $\Sigma_2$, $S_1$ rotates with respect to $S_f$ with an independent variable $\phi_1$ and $S_2$ rotates with respect to $S_f$ with a dependent variable $\phi_2$. Tooth surfaces $\Sigma_1$ and $\Sigma_2$ are assumed to be regular surfaces. At the common points of contact, the following conditions of contact are observed.
\[
\begin{align*}
R(u_1, v_1, \phi_1, u_2, v_2, \phi_2) &= r^{(1)}_f - r^{(2)}_f = 0 \\
N(u_1, v_1, \phi_1, u_2, v_2, \phi_2) &= n^{(1)}_f - n^{(2)}_f = 0
\end{align*}
\] (4)

where
\[
\begin{align*}
\begin{bmatrix}
r^{(1)}_f \\
n^{(1)}_f
\end{bmatrix} &= \mathbf{M}_f(\phi) \mathbf{r}(u, v) \\
\begin{bmatrix}
r^{(2)}_f \\
n^{(2)}_f
\end{bmatrix} &= \mathbf{M}_f(\phi) \mathbf{n}(u, v)
\end{align*}
\]

and
\[
\mathbf{M}_f(\phi)
\]

The matrix \( \mathbf{M}_f(\phi) \) denotes the coordinate transformation matrix of \( r \) \((i = 1, 2)\) and transforms the coordinates of \( r \) from \( S_i \) to \( S_f \). The matrix \( \mathbf{L}_f(\phi) \) represents the coordinate transformation matrix of \( n \) \((i = 1, 2)\) and transforms the coordinates of \( n \) from \( S_i \) to \( S_f \).

The conditions of contact represented in Eq. (4) are general conditions for any point of contact. Specifically, at the point \( A \), variables \( u_1, u_2, v_1, v_2, \phi_1 \) and \( \phi_2 \) are assumed to be
\[
\begin{align*}
u_1 &= u^{(1)}_1, v_1 = v^{(1)}_1, \phi_1 = \phi^{(1)}_1 \\
u_2 &= u^{(1)}_2, v_2 = v^{(1)}_2, \phi_2 = \phi^{(1)}_2
\end{align*}
\]

At point \( B \), variables \( u_1, u_2, v_1, v_2, \phi_1 \) and \( \phi_2 \) are assumed to be
\[
\begin{align*}
u_1 &= u^{(2)}_1, v_1 = v^{(2)}_1, \phi_1 = \phi^{(2)}_1 \\
u_2 &= u^{(2)}_2, v_2 = v^{(2)}_2, \phi_2 = \phi^{(2)}_2
\end{align*}
\]

Substituting the new variables in Eqs. (5) and (6) into Eq. (4) yields the following equations:
\[
\begin{align*}
\begin{bmatrix}
u^{(1)}_1 \\
v^{(1)}_1
\end{bmatrix} &= \mathbf{M}_f(\phi) \begin{bmatrix}
u^{(1)}_1 \\
v^{(1)}_1
\end{bmatrix} \\
\begin{bmatrix}
u^{(2)}_1 \\
v^{(2)}_1
\end{bmatrix} &= \mathbf{M}_f(\phi) \begin{bmatrix}
u^{(2)}_1 \\
v^{(2)}_1
\end{bmatrix} \\
\begin{bmatrix}
u^{(1)}_2 \\
v^{(1)}_2
\end{bmatrix} &= \mathbf{M}_f(\phi) \begin{bmatrix}
u^{(1)}_2 \\
v^{(1)}_2
\end{bmatrix} \\
\begin{bmatrix}
u^{(2)}_2 \\
v^{(2)}_2
\end{bmatrix} &= \mathbf{M}_f(\phi) \begin{bmatrix}
u^{(2)}_2 \\
v^{(2)}_2
\end{bmatrix}
\end{align*}
\] (7)

According to Eqs. (2) and (3), \( \phi^{(1)}_f \) and \( \phi^{(2)}_f \) can be expressed as functions of \( \xi \) as follows:
\[
\begin{align*}
\phi^{(1)}_f &= \left( N_1/N_2 \right) \phi^{(1)}_i - \xi \\
\phi^{(2)}_f &= \left( N_1/N_2 \right) \phi^{(2)}_i - \xi
\end{align*}
\] (8)

Since the period of motion function is \( (2\pi/N_i) \), \( \phi^{(1)}_f \) and \( \phi^{(2)}_f \) should comply with the following condition:
\[
\phi^{(1)}_f = \phi^{(1)}_i + (2\pi/N_1)
\] (9)

For a three dimensional problem, vector equations \( \mathbf{R}^{(1)} = 0 \) and \( \mathbf{R}^{(2)} = 0 \) have six independent scalar equations. Vector equations \( \mathbf{N}^{(1)} = 0 \) and \( \mathbf{N}^{(2)} = 0 \) have only four independent scalar equations since the length of any unit normal vector is equal to one. The total number of independent scalar equations is ten but there are only nine unknown parameters:
\[
\left\{u^{(1)}_1, v^{(1)}_1, u^{(1)}_2, v^{(1)}_2, u^{(2)}_1, v^{(2)}_1, u^{(2)}_2, v^{(2)}_2, \phi^{(1)}_f, \phi^{(2)}_f\right\}
\] (10)

Apparently, the number of independent equations is greater than the number of unknown variables. The number of equations minus the number of unknown variables is one, which means one design parameter, say \( \rho \), used to adjust the value of \( \xi \) can be treated as an unknown variable. After assigning one design parameter as an unknown variable, Eq. (7) forms a system of governing equations with ten unknown variables and ten independent scalar equations. By applying the Newton’s root finding method, the ten unknown variables can be determined. Feeding back the solution of \( \rho \) determined by the system of equations into the gear pair, the magnitude of parabolic-like transmission error will precisely comply with the desired value.

IV. APPLICATION TO A PAIR OF GEARS COMPOSED OF AN INVOLUTE GEAR AND A CIRCULAR-ARC GEAR

A. Setting up coordinate systems

A pair of gears composed of an involute gear and a circular-arc gear is used to show the mathematical procedures of controlling the magnitude of parabolic-like transmission error. The circular-arc gear is chosen to be input gear while the involute gear is chosen to be output gear. As shown in Fig. 3, coordinate systems \( S_i \) and \( S_f \) are applied to connect rigidly to the input and to the output gears, respectively. Coordinate system \( S_f \) is applied to connect rigidly to the generating racks of gears. When the coordinate system \( S_i \) rotates counterclockwise from the initial position to the angular displacement \( \phi_1 \), the coordinate system \( S_f \) moves leftward from the initial position to \( \rho \phi_1 \). When the
coordinate system $S_2$ rotates clockwise from the initial position to the angular displacement $\varphi_2$, the coordinate system $S_2$ moves leftward from the initial position to $\rho z \varphi$. 

**B. Creating mathematical model of gears**

As shown in Fig. 4, the shape of the generating rack of the involute gear is a straight line denoted by $L_2$. The shape of the generating rack of circular-arc gear is a circular arc denoted by $C_2$. The straight line $L_2$ and the circular arc $C_2$ are tangent at the origin of $S_2$. When coordinate systems move, straight line $L_2$ forms a family of curves in coordinate system $S_2$ and the envelope to the family of curves is the tooth shape of involute gear. Similarly, circular arc $C_2$ also forms a family of curves in $S_2$ and the envelope to the family of curves is the tooth shape of circular-arc gear.

The straight line $L_2$ is represented in $S_2$ by the following vector function:

$$\mathbf{r}_2(u_2) = [-u_2 \sin \alpha + m \tan \alpha, m - u_2 \cos \alpha, 0, 1]^T$$

In $S_2$, the family of curves formed by $L_2$ is determined by

$$\mathbf{r}_2(\varphi_2, u_2) = \mathbf{M}_2(\varphi_2) \mathbf{r}_2^{(2)}(u_2)$$

where

$$\mathbf{M}_2(\varphi_2) = \begin{bmatrix}
\cos \varphi_2 & -\sin \varphi_2 & 0 & -\rho_2 \varphi_2 \cos \varphi_2 + \rho_1 \sin \varphi_1 \\
\sin \varphi_2 & \cos \varphi_2 & 0 & -\rho_2 \varphi_2 \sin \varphi_2 - \rho_1 \cos \varphi_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Based on the theory of gearing [1], the necessary condition for the existence of envelope to the family of curves is

$$\left( \frac{\partial \mathbf{r}_2}{\partial u_2} \times \mathbf{k}_2 \right) \cdot \left( \frac{\partial \mathbf{r}_2}{\partial \varphi_2} \right) = -u_2 + m \sec \alpha - \rho \varphi_2 \sin \alpha = 0$$

According to Eq. (13), parameter $\varphi_1$ can be solved as an explicit function of $u_2$ as follows:

$$\varphi_1(u_2) = \frac{\csc \alpha (m \sec \alpha - u_2)}{\rho_1}$$

Substituting the explicit function $\varphi_1(u_2)$ into $\mathbf{r}_2(\varphi_2, u_2)$ yields a vector function $\mathbf{r}_2(u_2)$, which represents the shape of the involute gear generated by the straight line $L_2$. The involute gear’s unit normal vector is represented by

$$\mathbf{n}_2 = [-\cos(\alpha - \varphi_2), \sin(\alpha - \varphi_2), 0, 0]^T$$

The circular arc $C_2$ is represented in $S_2$ by the following vector function:

$$\mathbf{r}_2^{(1)}(u_1) = [\rho (\cos \alpha - \cos u_1), \rho (-\sin \alpha + \sin u_1), 0, 1]^T$$

The circular arc $C_2$ forms a family of curves in coordinate system $S_2$. In $S_2$, the family of curves formed by $C_2$ is determined by

$$\mathbf{r}_2(\varphi_1, u_1) = \mathbf{M}_2(\varphi_1) \mathbf{r}_2^{(1)}(u_1)$$

where

$$\mathbf{M}_2(\varphi_1) = \begin{bmatrix}
\cos \varphi_1 & \sin \varphi_1 & 0 & -\rho \varphi_1 \cos \varphi_1 + \rho_1 \sin \varphi_1 \\
-\sin \varphi_1 & \cos \varphi_1 & 0 & \rho \varphi_1 \sin \varphi_1 + \rho_1 \cos \varphi_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Based on the theory of gearing [1], the necessary condition for the existence of envelope to the family of curves is

$$\left( \frac{\partial \mathbf{r}_2}{\partial u_1} \times \mathbf{k}_2 \right) \cdot \left( \frac{\partial \mathbf{r}_2}{\partial \varphi_1} \right) = -\rho \varphi_1 \sin u_1 + \rho \sin(u_1 - \alpha) = 0$$

The parameter $\varphi_1$ can be solved as an explicit function of $u_1$ as follows:

$$\varphi_1(u_1) = \frac{\rho \left[ \csc u_1 \sin(u_1 - \alpha) \right]}{\rho_1}$$

Substituting $\varphi_1(u_1)$ into $\mathbf{r}_2(\varphi_1, u_1)$ produces a vector function $\mathbf{r}_2(u_1)$, which represents the shape of the circular-arc gear generated by the circular arc $C_2$. The circular-arc gear’s unit normal vector is as follows:

$$\mathbf{n}_1 = [-\cos(u_1 + \varphi_1), -\cos(u_1 + \varphi_1), 0, 0]^T$$

**C. Creating conditions of tooth contact**

The involute and the circular-arc gears are assembled in the fixed coordinate system $S_f$, as shown in Fig. 5. The circular-arc gear in the lower place is the active gear driving the passive involute gear in the upper place. When the active gear performs an angular displacement $\phi_1$, the passive gear will be forced to perform an angular displacement $\phi_2$. At the common points of contact of tooth faces of gears, the following two vector equations must be observed.
Fig. 5 Relation of motion of coordinate systems in meshing

\[
\begin{align*}
R(u_1, \phi_1, u_2, \phi_2) &= r^{(1)} - r^{(2)} = 0 \\
N(u_1, \phi_1, u_2, \phi_2) &= n^{(1)} - n^{(2)} = 0
\end{align*}
\]  
(21)

where
\[
\begin{align*}
r^{(1)} &= M_{r_1}(\phi_1) r_1(u_1) \\
r^{(2)} &= M_{r_2}(\phi_2) r_2(u_2) \\
n^{(1)} &= L_{n_1}(\phi_1) n_1(u_1) \\
n^{(2)} &= L_{n_2}(\phi_2) n_2(u_2)
\end{align*}
\]

D. Building the system of equations for the control of magnitude of parabolic-like transmission error

For contact point A, general parameters \( u_1, \phi_1, u_2, \phi_2 \) and \( \phi_1 \) in Eq. (11) are replaced by specific notations \( u_1^{(A)}, \phi_1^{(A)}, u_2^{(A)} \) and \( \phi_2^{(A)} \), respectively. Thus, contact conditions in Eq. (21) will produce the following two vector equations:

\[
\begin{align*}
R^{(A)}(u_1^{(A)}, \phi_1^{(A)}, u_2^{(A)}, \phi_2^{(A)}) &= 0 \\
N^{(A)}(u_1^{(A)}, \phi_1^{(A)}, u_2^{(A)}, \phi_2^{(A)}) &= 0
\end{align*}
\]  
(22)

Similarly, for contact B, universal parameters \( u_1, \phi_1, u_2, \phi_2 \) are substituted by specific symbols \( u_1^{(B)}, \phi_1^{(B)}, u_2^{(B)} \) and \( \phi_2^{(B)} \), respectively. Therefore, contact conditions will generate the following two vector equations:

\[
\begin{align*}
R^{(B)}(u_1^{(B)}, \phi_1^{(B)}, u_2^{(B)}, \phi_2^{(B)}) &= 0 \\
N^{(B)}(u_1^{(B)}, \phi_1^{(B)}, u_2^{(B)}, \phi_2^{(B)}) &= 0
\end{align*}
\]  
(23)

Parameters \( \phi_2^{(A)}, \phi_2^{(B)}, \phi_1^{(A)} \) and \( \phi_1^{(B)} \) are also constrained by

\[
\begin{align*}
\phi_2^{(A)} &= (N_1/N_2) \phi_1^{(A)} - \xi \\
\phi_2^{(B)} &= (N_1/N_2) \phi_1^{(B)} - \xi \\
\phi_1^{(B)} &= \phi_1^{(A)} + (2\pi/N_2)
\end{align*}
\]  
(24)

Equations (22)-(24) form a system of governing equations with six independent scalar equations and five unknown parameters as follows:

\[
\begin{align*}
\{u_1^{(A)}, u_2^{(A)}, u_1^{(B)}, u_2^{(B)}, \phi_1^{(A)}\}
\end{align*}
\]  
(25)

V. NUMERICAL EXAMPLE AND COMPUTER GRAPHICS

The numerical example corresponding to the mathematical models created in the previous section is provided here. The circular-arc gear has 21 teeth and the involute gear has 29 teeth. Gear module is 5 mm and pressure angle is 20 degrees. The desired value of magnitude of parabolic-like transmission error is inputted to the system of governing equations. The radius of circular arc \( C_1 \) is determined by the system of equations with Newton’s root finding method. Fig. 6 shows the relation between the magnitude of parabolic-like transmission error and the radius of circular arc \( C_1 \), the magnitude is increased from 5 arcsec to 40 arcsec. As the magnitude is increased, the radius of circular arc \( C_1 \) will decrease. It is noteworthy that their relation curve is a nonlinear one.

To verify the transmission error is a parabolic-like curve and the magnitude is equal to the pre-designed value, we set the desired magnitude to be 20 arcsec. The radius of circular arc \( C_1 \) is determined to be 535.7 mm. Based on the tooth contact analysis program created by the authors, the motion function for this specific case is obtained as shown in Fig. 7. Obviously, the transmission error curves are indeed parabolic-like ones and the magnitude is exactly 20 arcsec. Fig. 8 is the three dimensional solid models created by Autodesk® Mechanical Desktop software.

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VI. CONCLUSION

An intelligent method for the control of magnitude of parabolic-like transmission error for a pair of gears has been proposed. The value of the design parameter is determined automatically by the method without any artificial trial-and-error process. The proposed method is a general method since it can be applied to two or three dimensional gearing problems. A pair of external gears composed of an involute gear and a circular-arc gear is provided as an example. Numerical results show that the transmission errors are indeed parabolic-like ones. The relationship between the magnitude of parabolic-like transmission error and the radius of circular arc is a nonlinear curve.

REFERENCES