Highly nonlinear model in finance and convergence of Monte Carlo simulations

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Abstract

In this paper we consider the highly nonlinear model in finance proposed by Ait-Sahalia [Y. Ait-Sahalia, Testing continuous-time models of the spot interest rate, Rev. Finan. Stud. 9 (2) (1996) 385–426]. Both the drift and diffusion coefficients in this model do not obey the classical linear growth condition. To overcome the difficulties due to the highly nonlinear coefficients, we develop several new techniques to study the analytical properties of the model including the positivity and boundedness. In particular, we show that the Euler–Maruyama approximate solutions converge to the true solution in probability. The convergence result justifies clearly that the Monte Carlo simulations based on the Euler–Maruyama scheme can be used to compute the expected payoff of financial products e.g. options.

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1. Introduction

The interest rate is one of the most fundamental and important quantities in financial markets. Many mathematical models have been established to model it. One of the models is described by the stochastic differential equation (SDE)

\[ dx(t) = \lambda (\mu - x(t)) \, dt + \sigma x^\gamma (t) \, dw(t) \]  

(1.1)


On the other hand, some empirical studies show that the parameter \( \gamma > 1 \). For example, by the Generalized Method of Moments, Chan et al. [4] estimated \( \gamma = 1.449 \), while by the Gaussian estimation method, Nowman [16] showed \( \gamma = 1.361 \). It has therefore become more evident to consider \( \gamma > 1 \). It was in the same spirit that Ait-Sahalia in his seminal paper [1] proposed the following nonlinear SDE model

\[ dx(t) = (1 - \alpha_1 x^{-1}(t) - \alpha_0 + \alpha_1 x(t) - \alpha_2 x^2(t)) \, dt + \sigma x^\gamma (t) \, dw(t), \]  

(1.2)

where \( \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \sigma > 0, \gamma > 1 \).

The SDE (1.1) has been studied by many authors. In particular, when \( \gamma = 1/2 \), the solution of the SDE (1.1) is the well-known mean-reverting square root process [7]. It has been widely used to model volatility, interest rates and other financial quantities. Many authors (e.g. [10,12]) discussed its analytical properties, while Higham and Mao [9] examined the strong convergence of the Monte Carlo simulation. When \( \gamma \in [1/2, 1] \), Mao et al. [13] discussed its analytical properties as well as the strong convergence of numerical solutions. Recently, Wu et al. [18] study the model when \( \gamma > 1 \).

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However, there is a relatively little study on the SDE (1.2), in particular, numerical methods for it. We observe that both drift and diffusion coefficients of the SDE (2.1) do not obey the linear growth condition so we cannot apply the classical results (e.g., [12]) on the existence and uniqueness of the solutions, boundedness of the moment, the convergence of its Euler–Maruyama approximate solutions, and so on. This paper develops new techniques to overcome the difficulties arising from the highly nonlinear coefficients.

2. Global positive solutions

Throughout this paper, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (namely, it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Let \(w(t)\) be a scalar Brownian motion defined on the probability space. We consider the SDE (1.2) on \(\Omega, \mathcal{F}, \mathbb{P}\).

In order for a stochastic differential equation to have a unique global (i.e., no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see e.g., [12]). However, as pointed out before, the coefficients of Eq. (1.2) do not obey the linear growth condition, though it is locally Lipschitz continuous. We wonder if the solution of Eq. (1.2) may explode to infinity at any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see e.g., [12]). However, as pointed out before, the coefficients of Eq. (1.2) do not obey the linear growth condition, though it is locally Lipschitz continuous. We wonder if the solution of Eq. (1.2) may explode to infinity at a finite time. Furthermore, since Eq. (1.2) is used to model interest rate and other financial quantities, it is critical that the solution \(x(t)\) will never become negative. The following theorem reveals the existence of the unique global positive solution.

**Theorem 2.1.** Let the parameters \(\alpha = 1, \alpha_0, \alpha_1, \alpha_2, \sigma > 0\) and \(\gamma > 1\) (these will be the standing hypothesis of this paper). Then for any given initial value \(x(0) > 0\), there exists a unique global positive solution \(x(t)\) to Eq. (1.2) on \(t \geq 0\) with probability one.

**Proof.** In his original paper [1], Ait-Sahalia applied the well-know Feller test to prove the theorem. We will give an alternative proof that is more subtle and gives us more information which we will need later.

Since the coefficients of (2.1) satisfy the local Lipschitz condition in \((0, \infty)\), we can show by the standard truncation method (see e.g., [12]) that for any given initial value \(x(0) > 0\) there exists a unique maximal local solution \(x(t), t \in [0, \tau_x]\), where \(\tau_x\) is the stopping time of the explosion or first zero time. To prove our theorem, we need to show that \(\tau_x = \infty\) a.s. For a sufficiently large integer \(k > 0\), namely \(1/k < x(0) < k\), define the stopping time

\[
\tau_k = \inf\{t \in [0, \tau_x) : x(t) \notin (1/k, k]\},
\]

where throughout this paper we set \(\inf\emptyset = \infty\) (as usual \(\emptyset\) denotes the empty set). Clearly, \(\tau_k\) is increasing as \(k \to \infty\). Set \(\tau_\infty = \lim_{k \to \infty} \tau_k\), whence \(\tau_\infty \leq \tau_x\) a.s. If we can prove \(\tau_\infty = \infty\) a.s., then \(\tau_x = \infty\) a.s. and \(x(t) > 0\) a.s. for all \(t \geq 0\). In other words, to complete the proof all we need to show is that \(\tau_\infty = \infty\) a.s. To prove this, let us define a \(C^2\)-function \(V : (0, \infty) \to (0, \infty)\) by

\[
V(x) = \sqrt{x} - 1 - \frac{1}{2} \log(x). \tag{2.1}
\]

It is easy to see that \(V(\cdot) \geq 0\) and \(V(x) \to \infty\) as \(x \to \infty\) or \(x \to 0\). Applying the Itô formula yields

\[
dV(x(t)) = LV(x(t)) dt + \frac{1}{2} \left[ x^{-0.5} \sigma x^{\gamma}\right] dw(t), \tag{2.2}
\]

where \(LV : (0, \infty) \to \mathbb{R}\) is defined by

\[
LV(x) = \frac{1}{2} \left( x^{-0.5} - x^{-1}\right) \left( \alpha_1 x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^2\right) + \frac{1}{2} \sigma^2 x^{2\gamma} \left( -\frac{1}{4} x^{-1.5} + \frac{1}{2} x^{\gamma} \right). \tag{2.3}
\]

It is easy to see that \(LV\) is bounded, say by \(K_1\), namely

\[
LV(x) \leq K_1, \quad x \in (0, \infty). \tag{2.4}
\]

It therefore follows from (2.2) that, for any \(T > 0\),

\[
\mathbb{E}V(x(T \wedge \tau_k)) \leq V(x(0)) + K_1 T.
\]

Thus

\[
\mathbb{P}(\tau_k \leq T) \left[ V(1/k) \wedge V(k) \right] \leq \mathbb{E}V(x(T \wedge \tau_k)) \leq V(x(0)) + K_1 T,
\]

which gives

\[
\mathbb{P}(\tau_k \leq T) \leq \frac{V(x(0)) + K_1 T}{V(1/k) \wedge V(k)}. \tag{2.5}
\]

Therefore \(\mathbb{P}(\tau_k \leq T) \to 0\) since \(V(1/k) \wedge V(k) \to \infty\) as \(k \to \infty\). This implies \(\mathbb{P}(\tau_\infty = \infty) = 1\) as required. \(\square\)

It also follows from (2.5) directly the following corollary, which will be used later.
Corollary 2.2. For any given \( x(0) > 0, \varepsilon \in (0, 1) \) and \( T > 0 \), there is a sufficiently large integer \( k_0 = k_0(x(0), \varepsilon, T) \) such that
\[
\mathbb{P}(1/k_0 < x(t) < k_0 \text{ for all } 0 \leq t \leq T) \geq 1 - \varepsilon.
\]

3. Boundedness

For the interest rates and other asset prices, boundedness is a natural requirement. In this section, we will establish various boundedness for the solution to Eq. (1.2).

3.1. Stochastic boundedness

Corollary 2.2 shows that the solution of Eq. (1.2) will stay within \((1/k_0, k_0)\) during the time-interval \([0, T]\) with a large probability. However, the \( k_0 \) there will in general increase as \( T \) increases. In other words, the bound obtained there is not uniform in time. To get a uniform bound, let us present a useful lemma.

**Lemma 3.1.** There is a positive constant \( K_2 \), independent of the initial value \( x(0) \), such that the solution of Eq. (1.2) obeys
\[
\mathbb{E}\left[\sqrt{x(t)} - 1 - \frac{1}{2} \log(x(t))\right] \leq \sqrt{x(0)} - 1 - \frac{1}{2} \log(x(0)) + K_2. \quad \forall t \geq 0 \tag{3.1}
\]
and
\[
\limsup_{t \to \infty} \mathbb{E}\left[\sqrt{x(t)} - 1 - \frac{1}{2} \log(x(t))\right] \leq K_2. \tag{3.2}
\]

**Proof.** Define
\[
V_2(x, t) = e^t V(x), \quad (x, t) \in (0, \infty) \times (0, \infty),
\]
where \( V \) is the same as defined in the proof of Theorem 2.1. Let \( \tau_k \) be the same stopping time defined there too. Then, by the Itô formula,
\[
\mathbb{E}V_2(s(t \wedge \tau_k), t \wedge \tau_k) = V_2(x(0), 0) + \mathbb{E} \int_0^{t \wedge \tau_k} e^s \left[V(x(s)) + LV(x(s))\right] ds.
\]
Recalling (2.3), we see that there is a positive number \( K_2 \) such that
\[
V(x) + LV(x) \leq K_2, \quad \forall x \in (0, \infty).
\]
Therefore
\[
\mathbb{E}V_2(s(t \wedge \tau_k), t \wedge \tau_k) \leq V(x(0)) + K_2 e^t.
\]
Letting \( k \to \infty \) yields
\[
e^t \mathbb{E}V(x(t)) \leq V(x(0)) + K_2 e^t,
\]
which implies the required assertions (3.1) and (3.2). \( \square \)

Applying this lemma we can obtain the uniform bounds for the solutions.

**Theorem 3.2.** For any given \( x(0) > 0 \) and \( \varepsilon \in (0, 1) \), there is a number \( k_1 = k_1(x(0), \varepsilon) > 1 \) such that
\[
\mathbb{P}(1/k_1 < x(t) < k_1) \geq 1 - \varepsilon, \quad \forall t \geq 0.
\]

**Proof.** For any number \( k > 1 \), compute, by Lemma 3.1, that
\[
\mathbb{P}(x(t) \leq 1/k) \leq \mathbb{E}\left[I_{\{x(t) \leq 1/k\}} \frac{\sqrt{x(t)} - 1 - \frac{1}{2} \log(x(t))}{\sqrt{1/k} - 1 - \frac{1}{2} \log(1/k)}\right] \leq \frac{\sqrt{x(0)} - 1 - \frac{1}{2} \log(x(0)) + K_2}{\frac{1}{2} \log(k) - 1}
\]
for any \( t \geq 0 \), where throughout this paper \( I_A \) denotes the indicator function of set \( A \). Similarly,
\[
\mathbb{P}(x(t) \geq k) \leq \frac{\sqrt{x(0)} - 1 - \frac{1}{2} \log(x(0)) + K_2}{k - 1 - \frac{1}{2} \log(k)}.
\]
Hence
\[ P \left( \frac{1}{k} < x(t) < k \right) \geq 1 - \left[ \sqrt{x(0)} - 1 - \frac{1}{2} \log(x(0)) + K_2 \right] \left( \frac{1}{2} \log(k) - 1 + \frac{1}{k - 1} - \frac{1}{2} \log(k) \right). \]

This implies the assertion immediately. \( \square \)

The following theorem reveals that the solution will converge to a bounded interval with a large probability no matter where it starts.

**Theorem 3.3.** For any given \( \varepsilon \in (0, 1) \), there is a constant \( k_2 = k_2(\varepsilon) > 1 \) such that for every initial value \( x(0) \), the solution of Eq. (1.2) obeys
\[
\liminf_{t \to \infty} P \left( \frac{1}{k_2} < x(t) < k_2 \right) \geq 1 - \varepsilon.
\]

**Proof.** For any constant \( k > 1 \), we can show in the same way as in the previous proof that
\[
P \left( x(t) \leq \frac{1}{k} \right) \leq \frac{E \left[ \sqrt{x(t)} - 1 - \frac{1}{2} \log(x(t)) \right]}{\frac{1}{2} \log(k) - 1}
\]
and
\[
P \left( x(t) \geq k \right) \leq \frac{E \left[ \sqrt{x(t)} - 1 - \frac{1}{2} \log(x(t)) \right]}{k - 1 - \frac{1}{2} \log(k)}.
\]
Hence, by Lemma 3.1, we have
\[
\liminf_{t \to \infty} P \left( \frac{1}{k_2} < x(t) < k_2 \right) \geq 1 - K_2 \left( \frac{1}{2} \log(k) - 1 + \frac{1}{k - 1} - \frac{1}{2} \log(k) \right).
\]
This implies the assertion immediately. \( \square \)

### 3.2. Moment boundedness

We mainly focus on the boundedness of the first and the second moment.

**Theorem 3.4.** There is a positive constant \( K_4 \), independent of the initial value \( x(0) \), such that the solution of Eq. (1.2) obeys
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T E \left[ x^{-1}(t) + x(t) \right] dt \leq K_4.
\]  \tag{3.3}

**Proof.** We will use the same notations as in the proof of Theorem 2.1. Recalling (2.3), we see that there is a positive constant \( K_4 \) such that
\[
LV(x) + \frac{\alpha_2}{4} (x^{-1} + x) \leq \frac{\alpha_2 K_4}{4}, \quad \forall x \in (0, \infty).
\]
It then follows from (2.2) that
\[
\frac{\alpha_2}{4} E \int_0^{T \land \tau_k} \left[ x^{-1}(t) + x(t) \right] dt \leq V(x(0)) + \frac{\alpha_2 K_4 T}{4}.
\]
Letting \( k \to \infty \) and then applying the Fubini theorem, we get
\[
\frac{\alpha_2}{4} \int_0^T E \left[ x^{-1}(t) + x(t) \right] dt \leq V(x(0)) + \frac{\alpha_2 K_4 T}{4}.
\]
Dividing both sides by \( \alpha_2 T/4 \) and then letting \( T \to \infty \) we obtain the assertion. \( \square \)
Theorem 3.4 implies, for example,
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} x(t) \, dt \leq K_4.
\]
That is, the average in time of the first moment (the mean) of the solution is bounded. The theorem above is independent of parameter γ. Taking into account that γ ∈ (0, 1.5] in many situations (see e.g. [4,16]), we can have better results as described in the following two theorems.

**Theorem 3.5.** Assume that one of the following two conditions holds:

(i) γ ∈ (1, 1.5);
(ii) γ = 1.5 and 2α_2 > σ^2.

Then there is a positive constant K_5, independent of the initial value x(0), such that the second moment of the solution of Eq. (1.2) satisfies
\[
\mathbb{E} x^2(t) \leq x^2(0) e^{-t} + K_5, \quad \forall t \geq 0.
\]

**Proof.** The Itô formula shows that
\[
d[e^t x^2(t)] = e^t (x^2(t) + 2[\alpha_{-1} - \alpha_0 x(t) + \alpha_1 x^2(t) - \alpha_2 x^3(t)] + \sigma^2 x^2 \gamma (t)) \, dt \quad + \quad 2\sigma e^t x^{\gamma+1} (t) \, dw(t).
\]

Clearly, in either case (i) or (ii), there is a K_5 > 0 such that
\[
x^2 + 2[\alpha_{-1} - \alpha_0 x + \alpha_1 x^2 - \alpha_2 x^3] + \sigma^2 x^2 \gamma \leq K_5, \quad \forall x \in (0, \infty).
\]
It then follows from (3.5) (by the stopping time technique as in the proof of Lemma 3.1) that
\[
e^t \mathbb{E} x^2(t) \leq x^2(0) + K_5 e^t,
\]
which is the required assertion. □

**Theorem 3.6.** Assume that γ ∈ (1, 1.5). Then there is a positive constant K_6, independent of the initial value x(0), such that the solution of Eq. (1.2) satisfies
\[
\mathbb{E} x^{-1}(t) \leq x_1(0) e^{-t} + K_6, \quad \forall t \geq 0.
\]

**Proof.** Let y(t) = x^{-1}(t). The Itô formula shows that
\[
d[e^t y(t)] = e^t (y(t) - \alpha_{-1} y^3(t) + \alpha_0 y^2(t) - \alpha_1 y(t) - \alpha_2 + \sigma^2 y^{3-2\gamma}(t)) \, dt \quad + \quad \sigma e^t y^{\gamma+1} (t) \, dw(t).
\]

Clearly, there is a K_6 > 0 such that
\[
y - \alpha_{-1} y^3 + \alpha_0 y^2 - \alpha_1 y - \alpha_2 + \sigma^2 y^{3-2\gamma} \leq K_6, \quad \forall y \in (0, \infty).
\]
It then follows from (3.7) that
\[
e^t \mathbb{E} y(t) \leq x^{-1}(0) + K_6 e^t,
\]
which is the required assertion. □

When γ > 1.5, we do not have the nice properties as in the previous two theorems but we still have the following estimation.

**Theorem 3.7.** If γ > 1.5, then there is a positive constant K_7, independent of the initial value x(0), such that the solution of Eq. (1.2) obeys
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E} [x^{-2}(t) + x^2(t)] \, dt \leq K_7.
\]
Proof. As $\gamma > 1.5$, we can choose $\theta \in (0.5, 1)$ for $2\gamma + \theta - 2 > 2$. Define

$$V_3(x) = x^\theta - 1 - \theta \log(x), \quad x \in (0, \infty).$$

By the Itô formula,

$$0 \leq V_3(x(0)) + \mathbb{E} \int_0^{T \wedge \tau_k} LV_3(x(t)) \, dt,$$

where $\tau_k$ is the same stopping time defined in the proof of Theorem 2.1 while $LV_3 : (0, \infty) \to \mathbb{R}$ is defined by

$$LV_3(x) = \theta(x^{\theta-1} - x^{-1})(\alpha_1 x - \alpha_0 + \alpha_1 x - \alpha_2 x^2) + \frac{1}{2} \theta^2 x^{2\gamma} (-1 + x^{-2} + x^{-2}).$$

(3.10)

Recalling that $\theta \in (0.5, 1)$ and $2\gamma + \theta - 2 > 2$, we see that there is a positive constant $K_\gamma$ such that

$$LV_3(x) + \frac{1}{4} \alpha_1 (x^{-2} + x^2) \leq \frac{1}{4} K_\gamma, \quad x \in (0, \infty).$$

It therefore follows from (3.9) that

$$\mathbb{E} y(t) + \frac{1}{2} \alpha_1 \int_0^{T \wedge \tau_k} \left[ x^{-2} + x^2 \right] \, dt \leq V_3(x(0)) + \frac{1}{4} K_\gamma T.$$

From this, we can show the assertion in the same way as in the proof of Theorem 3.4. □

4. Pathwise asymptotic estimations

Let us now begin to discuss the more complicated pathwise properties of the solutions.

Theorem 4.1. If $\gamma \in (1, 1.5]$, then for any initial value $x(0) > 0$, the solution of Eq. (1.2) obeys

$$\liminf_{t \to \infty} \frac{\log x(t)}{\log t} \geq -1 \quad \text{a.s.}$$

(4.1)

Proof. Define $y(t) = x^{-1}(t)$. By the Itô formula, we have

$$dy(t) = \left[ -\alpha_1 y^3(t) + \alpha_0 y^2(t) - \alpha_1 y(t) - \alpha_2 + \sigma^2 y^{2\gamma} (t) \right] dt - \sigma y^{2\gamma-1}(t) \, dw(t).$$

(4.2)

For any sufficiently large integer $k$, define the stopping time

$$\rho_k = \inf\{ t \geq 0 : y(t) > k \}.$$

It then follows from (4.2) that, for any $T \geq 0$,

$$\mathbb{E} \left[ y(T \wedge \rho_k) + \frac{1}{2} \alpha_1 \int_0^{T \wedge \rho_k} y^3(t) \, dt \right] = y(0) + \mathbb{E} \int_0^{T \wedge \rho_k} \left[ -\frac{1}{2} \alpha_1 y^3(t) + \alpha_0 y^2(t) - \alpha_1 y(t) - \alpha_2 + \sigma^2 y^{2\gamma} (t) \right] \, dt.$$  

(4.3)

But there is clearly a positive number $K_8$ such that

$$\frac{1}{2} \alpha_1 y^3 + \alpha_0 y^2 - \alpha_1 y - \alpha_2 + \sigma^2 y^{2\gamma} \leq K_8, \quad y \in (0, \infty).$$

Hence

$$\mathbb{E} \left[ y(T \wedge \rho_k) + \frac{1}{2} \alpha_1 \int_0^{T \wedge \rho_k} y^3(t) \, dt \right] \leq y(0) + K_8 T.$$  

Letting $k \to \infty$ yields

$$\mathbb{E} \left[ y(T) + \frac{1}{2} \alpha_1 \int_0^T y^3(t) \, dt \right] \leq y(0) + K_8 T, \quad \forall T \geq 0.$$  

(4.4)

In particular, we have
\[
\mathbb{E} \int_{t}^{t+1} y^3(s) \, ds < \infty, \quad \forall t \geq 0.
\]

Having this, we can return to (4.2) to show, in the same way as above, that
\[
\mathbb{E} \int_{t}^{t+1} y^3(s) \, ds \leq \mathbb{E} y(t) + K_8, \quad \forall t \geq 0.
\]

(4.5)

We next observe that there is a \( K_9 > 0 \) such that
\[-\alpha_{-1} y^3 + \alpha_0 y^2 - \alpha_1 y - \alpha_2 + \sigma^2 y^{3-2\gamma} \leq K_9, \quad y \in (0, \infty).\]

It then follows from (4.2) that for any \( u \in [t, t+1] \),
\[y(u) \leq y(t) + K_9 - \int_{t}^{u} \sigma y^{2-\gamma}(s) \, dw(s).\]

This implies
\[
\mathbb{E} \left( \sup_{t \leq u \leq t+1} y(u) \right) \leq \mathbb{E} y(t) + K_9 + \sigma \mathbb{E} \left( \sup_{t \leq u \leq t+1} \left| \int_{t}^{u} y^{2-\gamma}(s) \, dw(s) \right| \right).
\]

(4.6)

Applying the Burkholder–Davis–Gundy inequality and the Jensen inequality, we compute that
\[
\mathbb{E} \left( \sup_{t \leq u \leq t+1} \left| \int_{t}^{u} y^{2-\gamma}(s) \, dw(s) \right| \right) \leq 3 \mathbb{E} \left( \int_{t}^{t+1} y^{2(2-\gamma)}(s) \, ds \right)^{\frac{1}{2}} \leq 3 \left( \mathbb{E} \left( \int_{t}^{t+1} y^{2(2-\gamma)}(s) \, ds \right) \right)^{\frac{1}{2}} \leq 3 \left( \mathbb{E} \left( \int_{t}^{t+1} y^3(s) \, ds \right) \right)^{\frac{2-\gamma}{3}}.
\]

By (4.5) and Theorem 3.6, we therefore see from (4.6) that there exists a constant \( K_{10} = K_{10}(x(0)) \) such that
\[
\mathbb{E} \left( \sup_{t \leq u \leq t+1} y(u) \right) \leq K_{10}.
\]

Let \( \varepsilon > 0 \) be arbitrary. By the Chebyshev inequality, we have
\[
\mathbb{P} \left\{ \sup_{k \leq t \leq k+1} y(t) > k^{1+\varepsilon} \right\} \leq \frac{K_{10}}{k^{1+\varepsilon}}. \quad k = 1, 2, \ldots.
\]

Applying the Borel–Cantelli lemma yields that for almost all \( \omega \in \Omega \),
\[
\sup_{k \leq t \leq k+1} y(t) \leq k^{1+\varepsilon}
\]

(4.7)

holds for all but finitely many \( k \). Hence, there exists a \( k_0(\omega) \), for almost all \( \omega \in \Omega \), for which (4.7) holds whenever \( k \geq k_0 \).

Consequently, for almost all \( \omega \in \Omega \), if \( k \geq k_0 \) and \( k \leq t \leq k+1 \),
\[
\frac{\log y(t)}{\log t} \leq \frac{(1+\varepsilon) \log k}{\log k} = 1 + \varepsilon.
\]

That is,
\[
\liminf_{t \to \infty} \frac{\log x(t)}{\log t} \geq -(1 + \varepsilon).
\]

Letting \( \varepsilon \to 0 \), we obtain the desired assertion (4.1). The proof is therefore complete. \( \Box \)

This theorem shows that for any \( \varepsilon > 0 \), there exists a positive random variable \( T_\varepsilon \) such that, with probability one,
\[
x(t) \geq t^{-(1+\varepsilon)}, \quad \text{for } \forall t \geq T_\varepsilon.
\]

(4.8)

In other words, with probability one, the solution will not decay faster than \( t^{-(1+\varepsilon)} \). The following theorem describes the growth property.

**Theorem 4.2.** Assume that one of the following two conditions holds:
(i) $\gamma \in (1, 1.5)$;
(ii) $\gamma = 1.5$ and $2\alpha_2 > \sigma^2$.

Then for any initial value $x(0) > 0$, the solution of Eq. (1.2) obeys
\[
\limsup_{t \to \infty} \frac{\log x(t)}{\log t} \leq 1 \quad \text{a.s.} \tag{4.9}
\]

**Proof.** The Itô formula shows that
\[
d[x^2(t)] = 2(\alpha - 1) - \alpha_0 x(t) + \alpha_1 x^2(t) - \alpha_2 x^3(t) + \sigma^2 x^{2\gamma}(t) dt + 2\sigma x^{\gamma+1}(t) dw(t). \tag{4.10}
\]
In either case (i) or (ii), we can choose $\delta > 0$ sufficiently small so that there is a $K_{11} > 0$ such that
\[
\delta x^3 + 2(\alpha - 1) - \alpha_0 x + \alpha_1 x^2 - \alpha_2 x^3 \leq K_{11}, \quad \forall x \in (0, \infty).
\]
It then follows from (4.10) that
\[
\delta E \int_t^{t+\delta} x^2(s) ds \leq E x^2(t) + K_{11}, \quad \forall t \geq 0. \tag{4.11}
\]

Note that for any $u \in [t, t + 1],
\[
x(u) = x(t) + \int_t^u (\alpha - 1 - 1^{-1}(s) - \alpha_0 + \alpha_1 x(s) - \alpha_2 x^2(s)) ds + \int_t^u \sigma x^{\gamma}(s) dw(s).
\]
But there is a $K_{12} > 0$ such that
\[-\alpha_0 + \alpha_1 x - \alpha_2 x^2 \leq K_{12}, \quad \forall x \in (0, \infty).
\]
Hence
\[
x(u) \leq x(t) + K_{12} + \alpha - 1 \int_t^u x^{-1}(s) ds + \int_t^u \sigma x^{\gamma}(s) dw(s).
\]
We therefore have
\[
E \left( \sup_{t \leq u \leq t+\delta} x(u) \right) \leq E x(t) + K_{12} + \alpha - 1 \int_t^{t+\delta} E x^{-1}(s) ds + E \left( \sup_{t \leq u \leq t+\delta} \int_t^u \sigma x^{\gamma}(s) dw(s) \right). \tag{4.12}
\]
But, by the Burkholder–Davis–Gundy inequality and the Jensen inequality, we compute that
\[
E \left( \sup_{t \leq u \leq t+\delta} \int_t^u \sigma x^{\gamma}(s) dw(s) \right)^{2} \leq 3 \left( E \int_t^{t+\delta} x^{2\gamma}(s) ds \right)^{2/3} \leq 3 \left( E \int_t^{t+\delta} x^{3}(s) ds \right)^{2/3} \leq 3 \left( E \int_t^{t+\delta} x^{3}(s) ds \right)^{2/3} \leq 3 \left( E \int_t^{t+\delta} x^{3}(s) ds \right)^{2/3}.
\]
By (4.11) and Theorems 3.5 and 3.6, we therefore obtain from (4.12) that there is a constant $K_{12} = K_{12}(x(0)) > 0$ such that
\[
E \left( \sup_{t \leq u \leq t+\delta} x(u) \right) \leq K_{12}, \quad t \geq 0. \tag{4.13}
\]
From this can we show the assertion (4.9) in the same way as in the proof of Theorem 4.1. 

This theorem shows that for any $\varepsilon > 0$, there exists a positive random variable $\bar{T}_\varepsilon$ such that, with probability one,
\[
x(t) \leq t^{(1+\varepsilon)}, \quad \text{for } \forall t \geq \bar{T}_\varepsilon. \tag{4.14}
\]
In other words, with probability one, the solution will not grow faster than $t^{(1+\varepsilon)}$. 

5. The Euler–Maruyama approximation

There is so far no explicit solution to Eq. (1.2). It is therefore critical to have numerical solutions. In this section we will discuss the Euler–Maruyama (EM) approximate solutions.

Let us begin with the definition of the discrete and continuous-time EM approximate solutions to Eq. (1.2). First of all, define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\begin{align*}
  f(0) &= 0 \quad \text{and} \quad f(x) = \alpha_1 x^{-1} - \alpha_0 + \alpha_1 x - \alpha_2 x^2 \quad \text{for } x \neq 0.
\end{align*}
\]

For a given fixed timestep \( \Delta \in (0, 1) \), set \( X_0 = x(0) > 0 \) and define

\[
X_{n+1} = X_n + f(X_n) \Delta + \sigma |X_n|^\gamma \Delta w_n
\]

(5.1)

for \( n = 0, 1, 2, \ldots \), where \( \Delta w_n = w((n+1) \Delta) - w(n \Delta) \). Introducing the continuous-time EM step process

\[
\bar{X}(t) = \sum_{n=0}^{\lfloor t/\Delta \rfloor} X_n I_{[n \Delta, (n+1) \Delta)}(t), \quad t \geq 0,
\]

(5.2)

we define the continuous-time EM solution

\[
X(t) = X_0 + \int_0^t f(\bar{X}(s)) \, ds + \sigma \int_0^t |\bar{X}(s)|^\gamma \, dw(s).
\]

(5.3)

It is useful to note that \( X(n \Delta) = \bar{X}(n \Delta) = X_n \) for all \( n \geq 0 \).

**Theorem 5.1.** For any \( T > 0 \),

\[
\lim_{\Delta \to 0} \left( \sup_{0 \leq t \leq T} |x(t) - X(t)| \right) = 0 \quad \text{in probability.}
\]

(5.4)

**Proof.** We divide the proof into two steps.

**Step 1.** For each sufficiently large \( k \), define the stopping time

\[
\rho_k = \inf \{ t \in [0, T] : X(t) \notin [1/k, k] \}.
\]

Let \( V \) be the same as defined by (2.1). By the Itô formula,

\[
\mathbb{E} V(X(T \wedge \rho_k)) = V(x(0)) + \int_0^{T \wedge \rho_k} \left[ V_x(X(s)) f(\bar{X}(s)) + \frac{1}{2} \sigma^2 V_{xx}(X(s)) |\bar{X}(s)|^{2\gamma} \right] \, ds.
\]

(5.5)

Recalling (2.3) and (2.4), we compute that for \( s \in [0, T \wedge \rho_k] \),

\[
\begin{align*}
V_x(X(s)) f(\bar{X}(s)) + \frac{1}{2} \sigma^2 V_{xx}(X(s)) |\bar{X}(s)|^{2\gamma} &= L V(x(s)) + V_x(X(s)) [f(\bar{X}(s)) - f(X(s))] + \frac{1}{2} \sigma^2 V_{xx}(X(s)) [||\bar{X}(s)||^{2\gamma} - |X(s)|^{2\gamma}] \\leq K_1 + c_k |X(s) - \bar{X}(s)|,
\end{align*}
\]

where throughout of this proof \( c_k \) denotes a positive constant dependent of only \( k \) which may change from line to line. We therefore obtain from (5.5) that

\[
\mathbb{E} V(X(T \wedge \rho_k)) \leq V(x(0)) + K_1 T + c_k \mathbb{E} \int_0^{T \wedge \rho_k} |X(s) - \bar{X}(s)| \, ds.
\]

(5.6)

For \( s \in [0, T \wedge \rho_k] \), let \([s/\Delta]\) denote the integer part of \( s/\Delta \). By definition (5.3),

\[
X(s) - \bar{X}(s) = f(X_{[s/\Delta]}) (s - [s/\Delta] \Delta) + \sigma |X_{[s/\Delta]}|^\gamma (w(s) - w([s/\Delta] \Delta)) \leq c_k \Delta + c_k |w(s) - w([s/\Delta] \Delta)|.
\]

(5.7)

We hence compute
Recalling that $V$ is the desired assertion (5.4).

Now, let $T^\wedge_k$ be the same as defined in the proof of Theorem 2.1. Let $\theta_k = \tau_k \wedge \rho_k$. It is now standard (see e.g. [11,12,14]) to show that there exists a positive constant $C = C(k, T)$ such that

$$
E \left[ \sup_{0 \leq t \leq T \wedge \rho_k} |x(t) - X(t)|^2 \right] \leq C \Delta.
$$

(5.10)

Now, let $\varepsilon, \delta \in (0, 1)$ be arbitrarily small. Set

$$
\bar{\Omega} = \left\{ \omega: \sup_{0 \leq t \leq T \wedge \rho_k} |x(t) - X(t)|^2 \geq \delta \right\}.
$$

Using (5.10), we compute

$$
\delta P(\bar{\Omega} \cap \{ \theta_k > T \}) \leq E \left[ \sup_{0 \leq t \leq T \wedge \rho_k} |x(t) - X(t)|^2 \right]
$$

$$
\leq E \left[ \sup_{0 \leq t \leq T \wedge \rho_k} |x(t) - X(t)|^2 \right]
$$

$$
\leq C \Delta.
$$

This, together with (2.5) and (5.9), implies

$$
P(\bar{\Omega}) \leq P(\bar{\Omega} \cap \{ \theta_k > T \} \cup \{ \theta_k \leq T \})
$$

$$
\leq P(\bar{\Omega} \cap \{ \theta_k > T \}) + P(\theta_k \leq T)
$$

$$
\leq C \Delta + \frac{2V(x(0)) + 2K_1 T + c_k \sqrt{T}}{V(1/k) \wedge V(k)}.
$$

(5.11)

Recalling that $V(1/k) \wedge V(k) \to \infty$ as $k \to \infty$, we can choose $k$ sufficiently large for

$$
\frac{2V(x(0)) + 2K_1 T}{V(1/k) \wedge V(k)} \leq \frac{\varepsilon}{2}
$$

and then choose $\Delta$ sufficiently small for

$$
\frac{C \Delta}{\delta} + \frac{c_k \sqrt{T}}{V(1/k) \wedge V(k)} < \frac{\varepsilon}{2}
$$

to obtain

$$
P(\bar{\Omega}) = P \left( \sup_{0 \leq t \leq T} |x(t) - X(t)|^2 \geq \delta \right) < \varepsilon,
$$

(5.12)

which is the desired assertion (5.4).

Step 2. Let $\tau_k$ be the same as defined in the proof of Theorem 2.1. Let $\theta_k = \tau_k \wedge \rho_k$. It is now standard (see e.g. [11,12,14]) to show that there exists a positive constant $C = C(k, T)$ such that

$$
E \left[ \sup_{0 \leq t \leq T \wedge \rho_k} \left| X(s) - \bar{X}(s) \right| ds \leq c_k T \Delta + \int_0^T E \left| w(s) - w([s/\Delta]) \right| ds
$$

$$
\leq c_k T \Delta + c_k \int_0^T E \left| w(s) - w([s/\Delta]) \right| ds
$$

$$
\leq c_k T \sqrt{\Delta}.
$$

Substituting this into (5.6) yields

$$
E V(X(T \wedge \rho_k)) \leq V(x(0)) + K_1 T + c_k T \sqrt{\Delta}.
$$

(5.8)

This implies

$$
P(\rho_k \leq T) \leq \frac{V(x(0)) + K_1 T + c_k T \sqrt{\Delta}}{V(1/k) \wedge V(k)}.
$$

(5.9)

Theorem 5.1 shows that the continuous-time EM approximate solution converges to the true solution uniformly in finite time interval in probability. However, the continuous-time EM solution is not possible to simulate in practice. It is therefore critically important to show the convergence for the discrete-time solution or the continuous-time step process (5.2). The following theorem describes this result.
Theorem 5.2. For any \( T > 0 \),
\[
\lim_{\Delta \to 0} \left( \sup_{0 \leq t \leq T} |x(t) - \tilde{X}(t)| \right) = 0 \quad \text{in probability.} \tag{5.13}
\]

This theorem follows easily from Theorem 5.1 and the following lemma.

Lemma 5.3. For any \( T > 0 \),
\[
\lim_{\Delta \to 0} \left( \sup_{0 \leq t \leq T} |X(t) - \tilde{X}(t)| \right) = 0 \quad \text{in probability.} \tag{5.14}
\]

Proof. We use the same notations as used in the proof of Theorem 5.1, but \( c_k \) will stand for a positive constant dependent of \( k \) and \( T \), and it may change from line to line. Recalling (5.7) we see easily that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X(t) - \tilde{X}(t)| \right) \leq c_k \Delta + c_k \mathbb{E} \left( \max_{0 \leq t \leq T} |w(k\Delta)| \right). \tag{5.15}
\]

By the Doob martingale inequality,
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |w(t) - w((t/\Delta)\Delta)| \right)^4 = \mathbb{E} \left( \sup_{0 \leq t \leq T} \sup_{k\Delta \leq t \leq (k+1)\Delta} |w(t) - w(k\Delta)| \right)^4 \\
\leq \sum_{k=0}^{[T/\Delta]} \mathbb{E} \left( \sup_{k\Delta \leq t \leq (k+1)\Delta} |w(t) - w(k\Delta)| \right)^4 \\
\leq \sum_{k=0}^{[T/\Delta]} \mathbb{E} |w((k+1)\Delta) - w(k\Delta)|^4 \leq 3 \sum_{k=0}^{[T/\Delta]} \Delta^4 \leq 3(T+1)\Delta.
\]

Hence, by the Hölder inequality, we have
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |w(t) - w((t/\Delta)\Delta)| \right) \leq \left[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |w(t) - w((t/\Delta)\Delta)| ^4 \right) \right]^{1/4} \leq \left[ 3(T+1)\Delta \right]^{1/4}.
\]

Substituting this into (5.15) gives
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X(t) - \tilde{X}(t)| \right) \leq c_k \Delta^{1/4}. \tag{5.16}
\]

For arbitrarily small constants \( \delta, \varepsilon \in (0, 1) \), set
\[
\bar{\omega} = \left\{ \omega : \sup_{0 \leq t \leq T} |X(t) - \tilde{X}(t)| \geq \delta \right\}.
\]

Then
\[
\delta \mathbb{P}(\bar{\omega} \cap \{ \rho_k > T \}) \leq \mathbb{E} \left( \max_{0 \leq t \leq T} \sup_{\rho_k \leq t} |X(t) - \tilde{X}(t)|^2 \right) \leq c_k \Delta^{1/4}. \tag{5.17}
\]

This, together with (5.9), yields that
\[
\mathbb{P}(\bar{\omega}) \leq \mathbb{P}(\bar{\omega} \cap \{ \rho_k > T \}) + \mathbb{P}(\rho_k \leq T) \leq \frac{c_k \Delta^{1/4}}{\delta} + \frac{V(x(0)) + K_1 T + c_k T \sqrt{\Delta}}{V(1/k) \wedge V(k)}. \tag{5.18}
\]

Choose \( k \) sufficiently large such that
\[
\frac{V(x(0)) + K_1 T}{V(1/k) \wedge V(k)} < \varepsilon \frac{\delta}{2}
\]
and then choose \( \Delta \) sufficiently small for
\[
\frac{c_k \Delta^{1/4}}{\delta} + \frac{c_k T \sqrt{\Delta}}{V(1/k) \wedge V(k)} < \varepsilon \frac{\delta}{2}.
\]

We therefore have
\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} |X(t) - \tilde{X}(t)|^2 \geq \delta \right) < \varepsilon, \tag{5.19}
\]

which is the desired assertion. \( \Box \)
6. Comments on option valuation

The convergence theory established in the previous section ensures that the EM method can be used to compute many financial quantities based on the asset price \( x(t) \) which obeys the SDE (1.2).

For example, for a European call option with the exercise price \( E \) at the expiry time \( T \), the payoff (or the value of the option) at the expiry time is

\[
C = E[(x(T) - E) \lor 0].
\]

Define its approximation

\[
C_\Delta = E[(\bar{X}(T) - E) \lor 0],
\]

where \( \bar{X}(t) \) is given by (5.2). Noting

\[
|C - C_\Delta| \leq E|x(T) - \bar{X}(T)|,
\]

we see, by an application of Theorem 5.2, that

\[
\lim_{\Delta \to 0} |C - C_\Delta| = 0.
\]

In the case where \( x(t) \) in (1.2) models the short-term interest rate dynamics, the price of a bond at the end of period is given by

\[
B(T) = E\left[\exp\left(-\int_0^T x(t) \, dt\right)\right].
\]

Using the step function \( \bar{X}(t) \) in (5.2), a natural approximation to \( B(T) \) is

\[
\bar{B}_\Delta(T) = E\left[\exp\left(-\int_0^T |\bar{X}(t)| \, dt\right)\right].
\]

Then

\[
\lim_{\Delta \to 0} |B(T) - \bar{B}_\Delta(T)| = 0.
\]

The proof can be found in [18].

As one more example, consider the case where Eq. (1.2) models a single barrier call option, which, at the expiry time \( T \), pays the European call value with the exercise price \( E \) if \( x(t) \) never exceeded the fixed barrier \( B \), and pays zero otherwise. The payoff at the expiry time is

\[
U = E[(x(T) - E)^+ I_{0 \leq x(T) \leq B, 0 \leq t \leq T}].
\]

Accordingly, define its approximation

\[
\bar{V}_\Delta := E[(\bar{X}(T) - E)^+ I_{0 \leq \bar{X}(T) \leq B, 0 \leq t \leq T}].
\]

based on the step process (5.2). Then

\[
\lim_{\Delta \to 0} |V - \bar{V}_\Delta| = 0.
\]

The proof can be found in [18] again.

All of these justify clearly that the EM method can be used to compute many financial quantities.

References